L_1 Regularized, Successive Greedy Solution of Heat Source Identification

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Outline

- Heat source identification
- Solution by L_1 minimization and Bregman iteration
- Strategies to reduce computational cost
- Adaptive solution with successive samples

Heat Source Identification

Consider the heat equation with sparse initial condition,

$$\begin{cases} u_t = u_{xx} + u_{yy} & t > 0 \\ u_0 = \sum_k c_k \delta(x - x_k, y - y_k) & t = 0,\ c_k > 0. \end{cases}$$

Suppose at time T we measure samples $f_{ij} = u(x_i, y_j)$.

Can we recover u_0 knowing that it is a sum of delta functions?

We can define a linear operator A_T such that $A_T[u_0] = u|_{t=T}$, so essentially the problem is inverting A_T (but A_T is ill-conditioned).

$L_0 \rightarrow L_1$

The goal is to use as few delta functions as possible to match up with the measurements, which is an L_0 minimization problem. As in compressed sensing, we approach the inverse by solving an L_1 minimization problem

$$\min_u |u|_1 \quad \text{subject to} \quad (Au)_i = f_i.$$

We shall use Bregman iteration for the minimization.

Bregman iterative method

The constrained problem:

$$\min_{u\in\mathbb{R}^n} |u|_1 \quad \text{subject to } Au = f.$$

The unconstrained problem:

$$\min_{u\in\mathbb{R}^n} E(u) = |u|_{\ell^1} + \lambda\|Au - f\|_{\ell^2}^2,$$

Algorithm:

1: Initialize: $k = 0$, $u^0 = \mathbf{0}$, $f^0 = f$.
2: while $\frac{\|Au-f\|_2}{\|f\|_2} >$ tolerance do
3: Solve $u^{k+1} \to \arg\min_u \; |u|_1 + \lambda\|Au - f^k\|_2^2$ by coordinate descent
4: $f^{k+1} \to f + (f^k - Au^{k+1})$
5: $k \to k+1$
6: end while

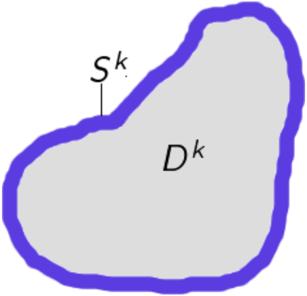
Preconditioner

Define $E = (AA^T + \epsilon)^{-1/2}$, instead, we solve

$$\min_u |u|_{\ell^1} \quad \text{subject to} \quad (EA)u = (Ef).$$

The preconditioner rescales the singular values such that the rows of EA are approximately orthogonal.

Support Restriction



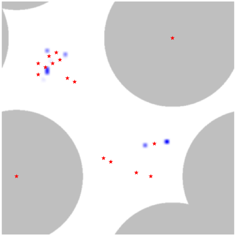
Define $D^k = \text{supp}(u^k)$, $S^k = D^k \bigcup \{\text{neighboring points of } D^k\}$.

$$u^{k+1} = \arg\min\{|u|_{\ell^1} + \lambda\|Au - f\|_2^2 : \text{supp}(u) \subset S^k\}.$$

Exclusion Region

Suppose the spike amplitudes are bounded from below by $\alpha_{\min} > 0$, then

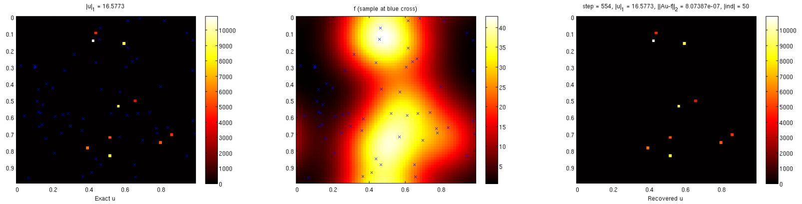
$$\exists k \text{ s.t. } A[\alpha_{\min}\delta_y](x_k) < f(x_k) \qquad \Longrightarrow \qquad u_0(y) = 0.$$



gray: excluded regions

Numerical Results

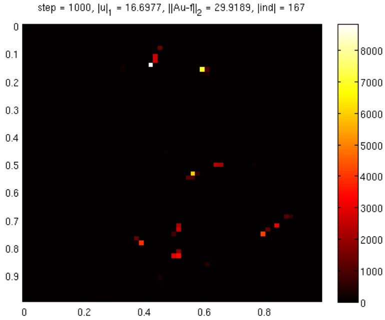
Exact u_0 | f | Recovery



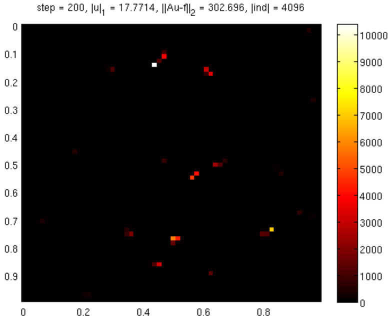
Heat source identification on the unit square with 60 measurements (runtime 7s).

Denoising

Recovery with 1% noise



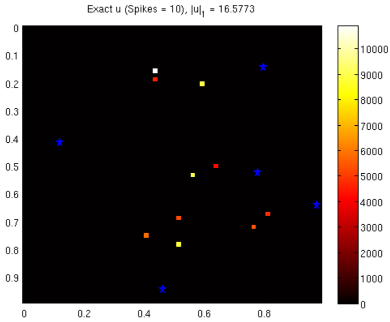
Recovery with 12% noise



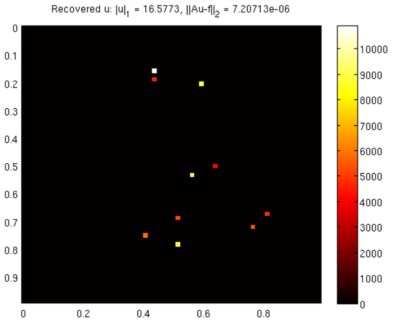
Recovery in the presence of noise.

Multiple Times

Exact u_0



Recovery



Observations sampled at 5 locations $\star$ at 10 different times uniform in $[0.002, 0.02]$ (runtime 24s).

Adaptive Solution

1. Solve the heat source identification problem with k samples;
2. Using the solution u_k to choose the $k+1$ sample;
3. Iterate.

Covering Region

Suppose the spike amplitudes are bounded from below by $\alpha_{\text{min}} > 0$, then

$$\exists k \text{ s.t. } A[\alpha_{\text{min}}\delta_y](x_k) \geq \text{threshold} \implies y \in \{\text{Covering region}\}.$$

We define a way to measure how much a point is covered by samples,

$$V(x) = A(\sum_j \delta_{x_j}).$$

The bigger $V(x)$ is , the more information available at x.

Refine locally or Explore further

Refine locally: If u_k varies significantly from u_{k-1}, then choose the next sampling location x_{k+1} by

$$x_{k+1} = \arg \max_{x : x \notin B_r(x_j)} |G_\sigma * u_k - G_\sigma * u_{k-1}|.$$

Explore further: Otherwise,

$$x_{k+1} = \arg \min_{x : x \notin B_r(x_j)} |V(x)|.$$

Numerical Experiment

Exact u_0

f

Numerical Experiment

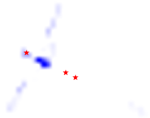
Recovery

Numerical Experiment

Recovery

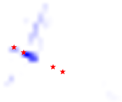
Numerical Experiment

Recovery



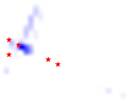
Numerical Experiment

Recovery



Numerical Experiment

Recovery

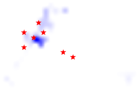


Numerical Experiment

Recovery

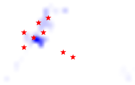
Numerical Experiment

Recovery



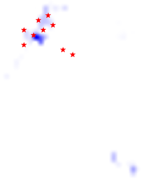
Numerical Experiment

Recovery



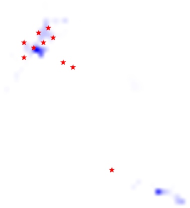
Numerical Experiment

Recovery



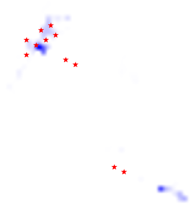
Numerical Experiment

Recovery



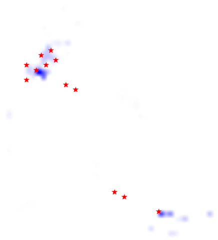
Numerical Experiment

Recovery



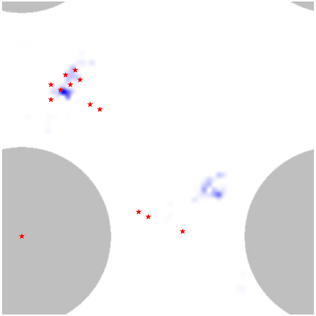
Numerical Experiment

Recovery



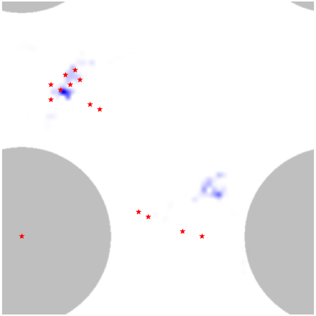
Numerical Experiment

Recovery



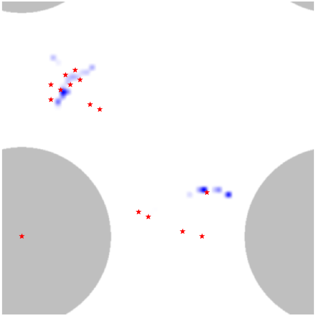
Numerical Experiment

Recovery



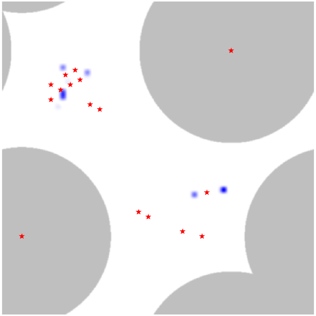
Numerical Experiment

Recovery



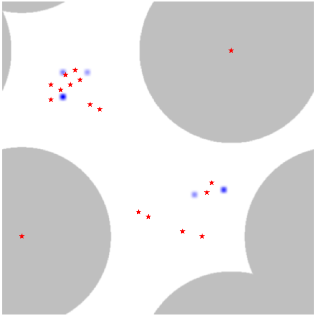
Numerical Experiment

Recovery



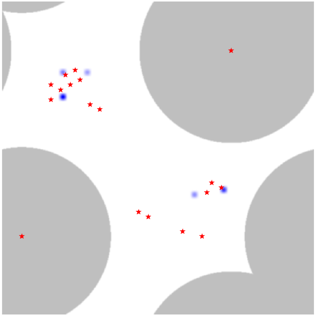
Numerical Experiment

Recovery



Numerical Experiment

Recovery



Done!

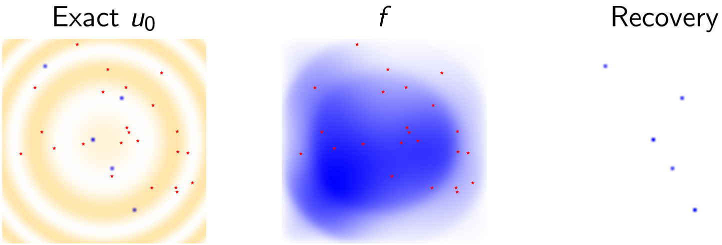
Inverting with Spatially-Varying Conductivity

Consider the equation with sparse initial condition,

$$\begin{cases} u_t = \operatorname{div}(a\nabla u) & x \in \Omega = (0,1)\times(0,1) \\ u = 0 & x \in \partial\Omega \\ u = \sum_k c_k \delta(x - x_k) & t = 0 \end{cases}$$

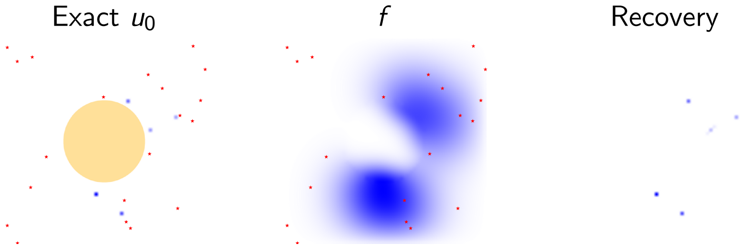
We sample u at time $T = 0.01$ and try to recover the initial condition using compressed sensing.

Numerical Results



The orange shows the distribution of $a(x)$. Here $a(x)$ is a positive smooth function.

Numerical Results



The orange circle indicates the distribution of $a(x)$, and $a(x) = 0.2$ inside of the circle and $a(x) = 1$ elsewhere.

Other Possibilities

- Can use adaptive method on

$$\partial_t u = \mathrm{div}(a \nabla u)$$

- Other linear PDEs, e.g. a process involving convection

$$\partial_t u = c \cdot \nabla u + \nu \Delta u$$

- Multigrid, adaptive grid refinement.

Questions?

Thanks for your attention!